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THE PROPAGATION OF A COMPLEX FRACTURE AREA. THE EXACT THREE-DIMENSIONAL SOLUTION*

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An analytic solution is constructed for the three-dimensional problem of the propagation of a rectangular fracture area on which a complex fracture process is given (cleavage with shear). A kinematic approach is used to describe the fracture process occurring at the discontinuity where the magnitude and direction of the displacement on the discontinuity is given on the whole fracture area as a boundary condition. Laplace and Fourier transforms and the Cagniard-Hoop method are used to determine the originals. The solution constructed extends the solution obtained earlier /1/ to the three-dimensional case.

At the present time research efforts in the area of theoretical modelling of fracture processes occurring in focal zones of tectonic earthquakes are directed towards the production of those models (describing the ripping open processes at a focus) which would allow a description of the singularities of high-frequency radiation in the best manner. Models should be noted in which the ripping open of a fault occurs in jumps (the barrier method) /2-4/, as well as the more general model based on a discrete jump-like ripping open of the fault along complex curvilinear trajectories /1, 5-7/.

The use of the extensively used Haskell computational model /8/ in seismic practice to analyse high-frequency radiation raises serious difficulties since the number of point sources taking part in the integration process increases catastrophically as the size of the fracture area grows. To eliminate these disadvantages, exact and compact analytic solutions must be constructed for, at least, the simplest models. A detailed survey of these papers is given in /2/, consequently, we note here only some fundamental results. Thus exact analytic solutions for a number of plane two-dimensional problems were constructed in /1, 6, 7, 9, 10/ (i.e., the width of the fault was assumed to be infinitely large), and a qualitative analysis is given of the singularities of seismic radiation generated both by single faults and complex systems of faults propagating at variable velocity along arbitrary curvilinear trajectories. Exact solutions have been constructed /11-14/ for circular and elliptical cracks, solutions were obtained /4/ of problems on the jumplike propagation of circular and annular dislocation faults. An analytic solution was obtained /15/ for a rectangular fault with pure shear and a constant function of the jump of the displacement on the fault.

The construct the general solution that takes account of the propagation of an arbitrary system of complex curvilinear discontinuities, it is necessary to have the solution of the problem of the propagating fracture area for which both a cleavage and shear component of the jump function for the displacement on the fault are present, as the fundamental solution. Consequently, the main purpose of this paper is the construction of an exact analytic solution extending the fundamental solution obtained in /1/ to the three-dimensional case.

1. Formulation of the problem. At a time $t = 0$ in a homogeneous isotropic elastic medium let a semi-infinite dislocation discontinuity (fault) with constant jump of the displacement $\mathbf{B}(B_x, B_y, B_z)$ originate along the positive z axis direction. One of the fault fronts starts to move at a constant velocity v_0 along the positive direction of the x axis, and the other is at rest and coincides with the position direction of the Oz axis. The $Oxyz$ coordinate system used to solve the problem of the fracture of a quadrant of space and the orientation of the fault is shown in Fig.1.

The equations of motion of the medium can be represented in the form of the following wave equations:

$$\Delta\Phi = \frac{1}{c_p^2}\frac{\partial^2\Phi}{\partial t^2}, \quad \Delta\Psi_i = \frac{1}{c_s^2}\frac{\partial^2\Psi_i}{\partial t^2} \quad (i = x, y, z) \tag{1.1}$$

(c_p and c_s are the longitudinal and transverse wave velocities, $c_p > c_s$, and Δ is the three-dimensional Laplace operator). The potentials Φ and Ψ_k are connected with the displacement vector $\mathbf{u}$ by the relationships

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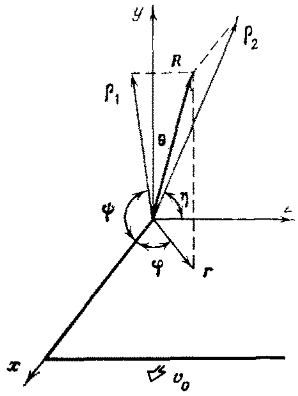

Fig.1

$$\mathbf{u} = \operatorname{grad} \Phi + \operatorname{rot} \Psi, \quad \operatorname{div} \Psi = 0 \tag{1.2}$$

The boundary and initial conditions are

$$y = 0, \tag{1.3}$$
$$[\mathbf{u}] = \mathbf{B} H(x) H(z) H(t - x/v_0)$$
$$t = 0, \quad \Phi = \partial\Phi/\partial t = 0, \tag{1.4}$$
$$\Psi = \partial\Psi/\partial t = 0$$

Here H is the Heaviside unit function and the square brackets denote the jump in the quantity enclosed in the brackets.

The displacements equal zero at infinity, i.e., the potentials Φ and Ψ and their space derivatives tend to zero as $R^2 = x^2 + y^2 + z^2 \to \infty$.

The stress tensor components σ_{ij} are associated with the displacement vector components by the relationships (ρ is the density of the medium)

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu (u_{i,j} + u_{j,i}) \quad (\lambda + \mu = \rho c_p^2, \quad \mu = \rho c_s^2)$$

A generalized dislocation fault with the displacement jump $[\mathbf{u}] = \mathbf{B}\,(B_x, B_y, B_z)$ can be represented in the form of the sum of a separation (normal) and shear fault. Then the boundary condition (1.3) can be replaced by the following boundary conditions:

for a pure separation fault

$$y = 0, \; u_y = {}^1/_2 B_y H(x) H(z) H(t - x/v_0), \; \sigma_{xy} = 0, \; \sigma_{zy} = 0 \tag{1.5}$$

for a pure shear fault

$$y = 0, \; u_x = {}^1/_2 B_x H(x) H(z) H(t - x/v_0), \; u_z = {}^1/_2 B_z H(x) \times H(z) H(t - x/v_0), \sigma_{yy} = 0 \tag{1.6}$$

We will construct the solution of the problem with the conditions (1.5). The solution of the problem with the condition (1.6) was obtained in /15/.

2. Construction of the formal solution.

Solutions of the wave Eqs.(1.1) satisfying conditions (1.4), (1.5) and the condition at infinity can be obtained by using a Laplace transform in the variable t and a double Fourier transform in the variables x and z:

$$f_L(x, y, z, k) = \int_0^\infty f(x, y, z, t) \exp(-kt)\, dt \tag{2.1}$$
$$f_F(\xi, y, \zeta, t) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} f(x, y, z, t) \exp[-i(\xi x + \zeta z)]\, dx\, dz$$

Later the subscript LF will be used to denote the result of both transformations in (2.1).

Applying (2.1) to the wave Eqs.(1.1), and taking account of the condition at infinity, we obtain

$$\Phi_{LF} = C \exp(-n_p y), \quad \Psi_{iLF} = C_i \exp(-n_s y) \tag{2.2}$$
$$n_{p,s} = (\xi^2 + \zeta^2 + k^2 c_{p,s}^{-2})^{1/2}$$

Applying the transformations (2.1) to the second equation and taking account of (2.2), we find

$$\Psi_{yLF} = i(\zeta C_x + \zeta C_z) n_s^{-1} \exp(-y n_s) \tag{2.3}$$

Applying the transformation (2.1) to the first equation in (1.2) and taking (2.2) and (2.3) into account, we obtain

$$u_{xLF} = i\xi C \exp(-yn_p) - \xi\zeta n_s^{-1} C_x \exp(-yn_s) + (n_s - \zeta^2 n_s^{-1}) C_z \exp(-yn_s)$$
$$u_{zLF} = i\zeta C \exp(-yn_p) + (\xi^2 n_s^{-1} - n_s) C_x \exp(-yn_s) + \xi\zeta n_s^{-1} C_z \exp(-yn_s)$$
$$u_{yLF} = -n_p C \exp(-yn_p) - i\zeta C_x \exp(-yn_1) + i\xi C_z \exp(-yn_s)$$

Then the boundary conditions (1.5) can be represented in the form

$$-2i\xi n_p C + 2\xi\zeta C_x - (2\xi^2 + k_s^2) C_z = 0 \tag{2.4}$$

$$-2i\zeta n_p C + (2\zeta^2 + k_s^2) C_x - 2\xi\zeta C_z = 0$$
$$-n_p C - i\zeta C_x + i\xi C_z = B_y [4\pi i k \zeta (i\xi + k v_0^{-1})]^{-1} \ (k_s = k c_s^{-1})$$

In the subsequent calculations it is sufficiently to limit oneself to a consideration of real values of k, as was shown in /16/ that according to the Lerch lemma even in this case the solution is determined uniquely.

Let us make the change of variables

$$\xi = -ikc_p^{-1}P_x, \ \zeta = -ikc_p^{-1}P_z$$

and we solve system (2.4) by the Cramer method. We obtain

$$C = -D[\beta_0^2 - 2P^2]\, m_p^{-1}, \ C_x = -2DP_z, \ C_z = -2DP_x, \tag{2.5}$$
$$P^2 = P_x^2 + P_z^2$$
$$D = B_y [4\pi k^3 c_p^{-2} P_z (P_x + \gamma)\beta_0^2]^{-1}, \ \gamma = c_p v_0^{-1}, \ \beta_0 = c_p c_s^{-1}$$

$$u_{iLF} = u_{iLF}^p + u_{iLF}^s = DF_i^p \exp(-ykc_p^{-1} m_p) + DF_i^s \exp(-ykc_p^{-1} m_s) \tag{2.6}$$

$$F_j^p = -P_j(\beta_0^2 - 2P^2)\, m_p^{-1}, \ F_j^s = 2P_j m_s, \ j = x, z \tag{2.7}$$
$$F_y^p = \beta_0^2 - 2P^2, \ F_y^s = 2P^2, \ m_p = (1 - P^2)^{1/2}, \ m_s = (\beta_0^2 - P^2)^{1/2}$$

The branch that has a positive real part is taken for the square roots $m_{p,s}$.

3. Finding the originals.

Applying the inverse Fourier transformation to the solutions (2.6), we obtain

$$u_{iL}^{p,s}(x, y, z, k) = -\frac{B_y}{8\pi^2 k \beta_0^2} \int_{-i\infty}^{i\infty} dP_x \int_{-i\infty}^{i\infty} dP_z \frac{F_i^{p,s}(P_x, P_z)}{P_z(P_x + \gamma)} \exp\{kc_p^{-1}[\mathbf{pr} - y m_{p,s}]\} \tag{3.1}$$

$$\mathbf{pr} = P_x x + P_z z = c_p k^{-1} i(\xi x + \zeta z), \ r^2 = x^2 + z^2$$

We use the Cagniard-Hoop method /16, 17/ to integrate (3.1) and obtain the solution in the time domain.

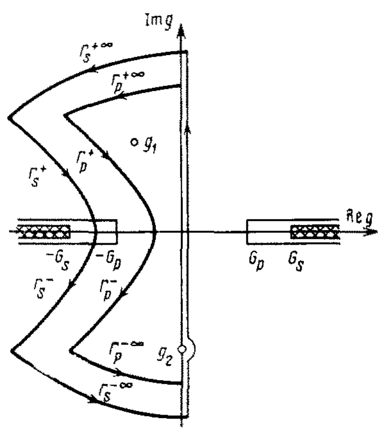

Fig.2

We apply the following modified de Hoop transformation

$$P_x = g\cos\varphi - iw\sin\varphi, \ P_z = g\sin\varphi + iw\cos\varphi \tag{3.2}$$

$$m_p = (w_p^2 - g^2)^{1/2}, \ m_s = (w_s^2 - g^2)^{1/2}, \ w_p^2 = w^2 + 1, \tag{3.3}$$
$$w_s^2 = w^2 + \beta_0^2$$

We note that to use this method successfully the parameter k must only be in the exponent of the component. The parameter k is present in (3.1) in the denominator of the factor in front of the integral. To isolate it, it is sufficient to consider later the velocities $\dot{u}_{iL}^{p,s}$ rather than the displacements $u_{iL}^{p,s}$. In conformity with the properties of the Laplace transform

$$\dot{u}_{iL}(\mathbf{R}, k) = k u_{iL}(R, k) - u_i(R, 0)$$

where $u_i(R, 0) = 0$ agrees with the initial conditions. We then obtain from (3.1)

$$\dot{u}_{iL}^{p,s}(\mathbf{R}, k) = -\frac{B_y i}{8\pi^2\beta_0^2} \int_{-\infty}^{\infty} J\, dw \tag{3.4}$$

$$J = \int_{-i\infty}^{i\infty} \frac{F_i^{p,s}(g, w)}{P_z(P_x + \gamma)} \exp[kc_p^{-1}(gr - y m_{p,s})]\, dg$$

Here $\mathbf{R}$ is a radius-vector with the direction cosines

$$\nu_x = \sin\theta\cos\varphi, \ \nu_y = \cos\theta, \ \nu_z = \sin\varphi\cos\theta, \ \nu_x^2 + \nu_y^2 + \nu_z^2 = 1$$

(the angles φ, θ are shown in Fig.1).

We note that the singularities in the integrands in (3.4) are the bifurcation point $G_{p,s}$ and simple poles g_1 and g_2 (Fig.2) determined from the conditions

$$G_{p,s} = \pm(w^2 + \beta_{p,s}^2)^{1/2}, \quad \beta_p = 1, \quad \beta_s = \beta_0 \tag{3.5}$$

$$g_1 = -iw \operatorname{ctg} \varphi, \quad g_2 = -\gamma/\cos\varphi + iw \operatorname{tg}\varphi. \tag{3.6}$$

To evaluate the inner integral in (3.4), we go from integration over the imaginary axis to integration over the closed contour and we use the residue theorem. Then

$$J = -\int\limits_{\Gamma_{p,s}} \frac{F_i^{p,s}(g, w)}{P_z(P_x+\gamma)} \exp[kc_p^{-1}(gr - ym_{p,s})]\,dg + 2\pi i\left[\frac{F_i^{p,s}(g_1, w)}{(P_x+\gamma)\sin\varphi}\exp[kc_p^{-1}(g_1r - ym_{p,s})] + \frac{F_i^{p,s}(g_2, w)}{P_z\cos\varphi}\exp[kc_p^{-1}(g_2r - ym_{p,s})]\right] \tag{3.7}$$

where $\Gamma_{p,s}$ are the contours shown in Fig.2.

Substituting (3.7) into (3.4), we obtain

$$u_{iL}^{p,s}(R, k) = \frac{iB_y}{8\pi^2\beta_0^2} J_\Gamma^{p,s} + \frac{B_y}{4\pi\beta_0^2}[J_1^{p,s} + J_2^{p,s}] \tag{3.8}$$

where $J_\Gamma^{p,s}$, $J_1^{p,s}$, $J_2^{p,s}$ are integrals taken from the first, second, and third terms on the right-hand side of (3.7), respectively.

To evaluate the integrals $J_1^{p,s}$ we deform the contour of integration in the w plane into a Cagniard contour, which is defined parametrically by the equation (t is a real parameter)

$$t = -c_p^{-1}(g_1r - ym_{p,s}) \tag{3.9}$$

Substituting the first relationship of (3.6) and (3.3) into (3.9), we obtain

$$tc_p - iw\frac{\nu_x}{\nu_z} r = (w_{p,s}^2 - g^2)^{1/2}\, y, \quad \beta_p = 1, \quad \beta_s = \beta_0 \tag{3.10}$$

Taking into account that

$$r\sin\varphi = R\nu_z, \quad y = R\nu_y, \quad \nu_z = \sin\varphi\sqrt{\nu_x^2 + \nu_z^2}$$

and solving (3.10) for w, we obtain

$$w = -i\tau\frac{\nu_x \sin\varphi}{\alpha_z^2} \pm \frac{\nu_y \sin\varphi\sqrt{\tau^2 - \alpha_z^2\beta_{p,s}^2}}{\alpha_z^2} \tag{3.11}$$

$$\left(\tau = \frac{tc_p}{R}, \quad \alpha_z^2 = 1 - \nu_z^2\right)$$

We convert (3.11) by going over to the ρ_1, ψ coordinate system (Fig.1). Since $\rho_1^2 = R^2\alpha_z^2$, $R\nu_x = \rho_1\cos\psi$, $R\nu_y = \rho_1\sin\psi$, (3.11) can be converted to the form

$$w = -i\tau_1\sin\varphi\cos\psi \pm T_1^{p,s}\sin\psi\sin\varphi \tag{3.12}$$

$$\tau_1 = tc_p/\rho_1, \quad T_1^{p,s} = (\tau_1^2 - \beta_{p,s}^2)^{1/2}, \quad \tau_1 > \beta_{p,s}$$

To be specific, we select the plus sign before the second component in (3.12), i.e., we consider the upper sheet of the Riemann surface.

Having determined dw from (3.12), we obtain that

$$J_1^{p,s} = \frac{c_p}{\rho_1}\int\limits_{-\infty}^{\infty} \frac{F_i^{p,s}(g_1, w)[\tau_1\sin\psi - iT_1^{p,s}\cos\psi]}{(P_x+\gamma)T_1^{p,s}}\exp(-kt)\,dt \tag{3.13}$$

Taking account of the first relations in (3.6) and (3.12), we obtain from (3.2) and (3.3)

$$P_x = -\tau_1\cos\psi - iT_1^{p,s}\sin\psi, \quad P_z = 0 \tag{3.14}$$

$$m_{p,s} = \tau_1\sin\psi - iT_1^{p,s}\cos\psi$$

Then substituting (3.14) into (2.7), it can be seen that the integrands in (3.13) have an even real part and an odd imaginary part. Therefore, (3.13) can be represented in the form

$$J_1^{p,s} = \frac{2c_p}{\rho_1}\operatorname{Re}\left[\int\limits_0^{\infty}\frac{F_i^{p,s}(g_1, w)\, m_{p,s}}{(P_x+\gamma)T_1^{p,s}}\exp(-kt)\,dt\right] \tag{3.15}$$

This equation is a direct Laplace transform of the integrand in the exponential. According to rules of the operational calculus, we find from (3.15)

$$J_1^{p,s} = \frac{2c_p}{\rho_1} \operatorname{Re}\left[\frac{F_i^{p,s}(g_1, w)\, m_{p,s}}{(P_x + \gamma)\, T_1^{p,s}}\right] H(z) H(\tau_1 - \beta_{p,s}) \tag{3.16}$$

The functions $H(z)$ and $H(\tau_1 - \beta_{p,s})$ characterise the domain of existence of the integral $J_1^{p,s}$. The functions $H(\tau_1 - \beta_{p,s})$ govern the moments of P-wave and S-wave arrival at the point of observation while the functions $H(z)$ indicate that the integrals $J_1^{p,s}$ differ from zero only for $z > 0$ since the pole g_1 will be within the closed contour only for $\sin\varphi > 0$ (or $z > 0$). For $z < 0$ the pole g_1 will lie outside the contour and it should be bypassed to the left.

To evaluate the integrals $J_2^{p,s}$ we define the Cagniard contour in the form

$$t = -c_p^1 (g_2 r - y m_{p,s}) \tag{3.17}$$

Substituting the second relationship of (3.6) into (3.17), and performing an analogous series of calculations, we obtain

$$w = -i \sin\varphi \cos\varphi \left[\frac{\gamma \sin^2\eta}{\cos\varphi} - \frac{\cos\eta}{\sin\varphi}(\tau_2 - \gamma\cos\eta \operatorname{tg}\varphi)\right] \pm T_2^{p,s} \sin\eta\cos\varphi \tag{3.18}$$

where

$$\tau_2 = \left(t - \frac{x}{v_0}\right)\frac{c_p}{\rho_2}, \quad \rho_2 = \frac{z}{\cos\eta} = \frac{x\sin\varphi}{\cos\varphi\cos\eta}, \quad T_2^{p,s} = (\tau_2^2 + \gamma^2 - \beta_{p,s}^2)^{1/2}$$

(ρ_2, η are cylindrical coordinates around the x axis (Fig.1)).

To determine the domain of existence of the integrals $J_2^{p,s}$ it is necessary to examine the conditions under which the pole g_2 will be within the closed contour. Substituting (3.3) into (3.17) and solving the quadratic equation obtained for g we obtain

$$g = c_p R^{-2}\left[-tr \pm iy\sqrt{t^2 - R^2 c_p^{-2} w_{p,s}^2}\right]$$

This equation determines the branch of the hyperbola in the complex g plane; the asymptotes of the hyperbola determine the angle $\alpha = \pm\operatorname{arctg} y/r$. The real part of this expression is always negative. Then the pole $g_2 = -\gamma/\cos\varphi + iw\operatorname{tg}\varphi$ will be within the closed contour under the following conditions:

$$\begin{aligned}
&\operatorname{Re} g_2 < 0, \quad \text{or} \quad x > 0\\
&\operatorname{Re} g_2 > \operatorname{Re} g, \quad \text{or} \quad t > R/v_0 v_x = t_0\\
&\operatorname{Im} g_2 > \operatorname{Im} g, \quad \text{or} \quad w\operatorname{tg}\varphi > y c_p R^{-2}\sqrt{t^2 - R^2 C_p^{-2} w_{p,s}^2}
\end{aligned}$$

It follows from the last two relationships that

$$w^2 > \frac{(R^2\gamma^2 - x^2\beta_{p,s}^2)\, y^2}{r^2 (y^2 + z^2)} = w_0$$

Therefore, we have obtained that depending on the value of the rate of ripping open the fault, the following domains of definition of the solution are possible: for $x > 0$ and $\gamma\beta_{p,s}^{-1} < xR^{-1}$ the pole g_2 will be within the contour for $w \in [0, \infty[$; for $x > 0$ and $\gamma\beta_{p,s}^{-1} > xR^{-1}$ the pole g_2 will be within the contour for $w \in [w_0, \infty[$; for $x < 0$ the pole will be outside the contour.

For the fault ripped open at the rate $v_0 > c_p$ the integrals $J_2^{p,s}$ will exist within the domain (Fig.3a) whose boundaries are determined from the relationships

$$\begin{aligned}
&t = t_p^* = v_0^{-1}(\sqrt{\gamma^{-2} - 1}\,\rho_2 + x) \quad \text{and} \quad t = t_0 \text{ for } P\text{-waves}\\
&t = t_s^* = v_0^{-1}(\sqrt{\beta_0^2\gamma^{-2} - 1}\,\rho_2 + x) \quad \text{and} \quad t = t_0 \text{ for } S\text{-waves}
\end{aligned}$$

For $c_p > v_0 > c_s$ the solution will exist within the domain (Fig.3b) whose boundaries are determined from the relationships

$$t = t_0 \text{ for } P\text{-waves}; \; t = t_s^* \text{ and } t = t_0 \text{ for } S\text{-waves}$$

For $c_p > c_s > v_0$ the integrals $J_2^{p,s}$ will differ from zero within the domain $t = t_0$ (Fig. 3c).

We will henceforth limit ourselves to considering the case when $c_p > c_s > v_0$.

The following the same procedures as when finding $J_1^{p,s}$, we obtain

$$J_2^{p,s} = \frac{2c_p}{\rho_2}\operatorname{Re}\left[F_i^{p,s} m_{p,s}/P_z T_2^{p,s}\right] H(\tau_2 - \tau_0) \tag{3.19}$$

To determine the functions $F_i^{p,s}$ in (3.19) we have the following expressions obtained by taking account of (3.18), (3.2), (3.3) and the second relationship in (3.6):

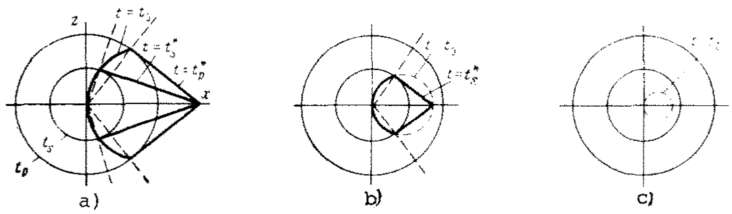

Fig.3

$$P_x = -\gamma, \quad P_z = -\tau_2 \cos\gamma + iT_2^{p,s} \sin\eta$$
$$m_{p,s} = \tau_2 \sin\eta + iT_2^{p,s} \cos\eta$$

The function $H(\tau_2 - \tau_0)$, $\tau_0 = \gamma\rho_2/|x|$, which determines the condition for finding the pole g_2 within the Cagniard contour is present in (3.19).

The Cagniard contour for evaluating $J_\Gamma^{p,s}$ is given by the equation

$$t = -c_p^{-1}(gr - ym_{p,s}), \quad \tau = -g\sin\theta + m_{p,s}\cos\theta \tag{3.20}$$

Substituting (3.3) into (3.20) and solving for g, we obtain

$$g(\tau, w) = (w_{p,s}^2 - \tau^2)^{1/2}\cos\theta - \tau\sin\theta \tag{3.21}$$

Having determined g from (3.3) and substituting the result into (3.20), and then solving the equation for $m_{p,s}$, we have

$$m_{p,s} = (w_{p,s}^2 - \tau^2)^{1/2}\sin\theta + \tau\cos\theta \tag{3.22}$$

The configuration of the Cagniard contour is a function of the real parameter $\tau = tc_pR^{-1}$ and the real variable w. For $\tau < (w^2 + \beta_{p,s}^2)^{1/2}$ the contour agrees with the axis $\operatorname{Re} g$, and this section makes no contribution to the integral (it is not considered in Fig.2). For $\tau > (w^2 + \beta_{p,s}^2)^{1/2}$ the square root becomes imaginary and the Cagniard contour splits into two parts Γ^+ and Γ^- (Fig.2) depending on the sign of the square root

$$J_\Gamma^{p,s} = I_\Gamma^{+\infty} + I_\Gamma^{+} + I_\Gamma^{-} + I_\Gamma^{-\infty} \tag{3.23}$$

$$I_\Gamma^{\pm} = \int_{-\infty}^{\infty} dw \int_{(w^2+\beta_{p,s}^2)^{1/2}}^{\infty} d\tau \left[\frac{F_i^{p,s}}{P_z(P_x+\gamma)}\frac{dg}{d\tau}\right]^{\pm} \exp(-kRc_p^{-1}\tau)$$

The integrals $I_\Gamma^{+\infty}$, $I_\Gamma^{-\infty}$ are evaluated along the contour $\Gamma_{p,s}^{\pm\infty}$ and tend to zero as $|g| \to \infty$ according to the Jordan lemma. The plus and minus superscripts correspond to the integrals taken along Γ^+ or Γ^-.

We find

$$dg/d\tau = -m_{p,s}(w_{p,s}^2 - \tau^2)^{-1/2}, \quad |w| < (\tau^2 - \beta_{p,s}^2)^{1/2} = T_{p,s}$$

from (3.21) and (3.22) and because $(w^2 + \beta_{p,s}^2)^{1/2} < \tau$ in (3.23).

Since w is a real parameter, then $\tau > \beta_{p,s}$. Going over from τ to $t\,(\tau R = tc_p)$, we obtain

$$I_\Gamma^{\pm} = -\int_{-T_{p,s}}^{T_{p,s}} dw \int_{Rc_p^{-1}\beta_{p,s}}^{\infty} [M_i^{p,s}]^{\pm} \exp(-kt)\,dt$$

$$M_i^{p,s} = F_i^{p,s} m_{p,s}/P_z(P_x+\gamma)(w^2 - T_{p,s}^2)^{1/2}$$

or

$$I_\Gamma^{\pm} = -c_pR^{-1}\int_{-T_{p,s}}^{T_{p,s}} dw \int_0^{\infty} [M_i^{p,s}]^{\pm} H(\tau - \beta_{p,s}) \exp(-kt)\,dt \tag{3.24}$$

Taking into account that the inner integral in (3.24) is a direct Laplace transform of the integrand in the exponential, it can be written in the domain of originals

$$I_\Gamma^{\pm} = -\frac{c_p}{R}\int_{-T_{p,s}}^{T_{p,s}} dw\,[M_i^{p,s}]^{\pm} H(\tau - \beta_{p,s}) \tag{3.25}$$

Then if the w plane is slit as shown in Fig.4, then by taking the positive value of the root $(w^2 - T_{p,s}^2)^{1/2}$ and taking account of (3.23) and (3.25) we can write on the upper edges of the slits

$$J_{\Gamma}^{p,s} = -\frac{c_p}{R} \oint_{\Gamma_{p,s}} M_i^{p,s} H(\tau - \beta_{p,s})\, dw \qquad (3.26)$$

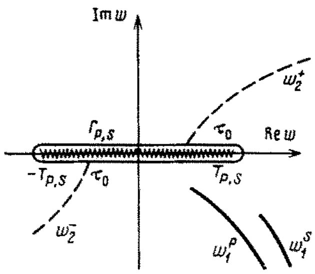

Fig.4

where $\Gamma_{p,s}$ is the closed contour around the slits. Using the theorem of residues to evaluate (3.26), we have

$$J_{\Gamma}^{p,s} = 2\pi i c_p R^{-1} (\mathrm{Res}_1 + \mathrm{Res}_2 + \mathrm{Res}_\infty)$$

where Res_1, Res_2, Res_∞ are the residues at the poles $P_z = 0$, $P_x = -\gamma$ and infinity, respectively.

To evaluate Res_1 we obtain from the second relationship in (3.2) and (3.21)

$$P_z = (w_1^2 - \tau^2 + \beta_{p,s}^2)^{1/2} \cos\theta \sin\varphi - r \sin\theta \sin\tau + i w_1 \cos\varphi = 0 \qquad (3.27)$$

Taking into account that $\nu_x = \sin\theta\cos\varphi$, $\nu_y = \cos\theta$, $\nu_z = \sin\theta \sin\varphi$, we have from (3.27)

$$w_{1p,s} = \{-i\tau\cos\varphi \pm \nu_y \sin\varphi\, [\tau^2 - \alpha_z^2\beta_{p,s}^2]^{1/2}\}\alpha_z^{-2} \qquad (3.28)$$

For a single-valued selection of the sign in (3.28) we consider the root to take a positive value on the upper sheet of the Riemann surface. We then obtain

$$\mathrm{Res}_{1p,s} = F_i^{p,s} m_{p,s} [(P_x + \gamma) T_1^{p,s} \alpha_z]^{-1} H(\tau_1 - \beta_{p,s}\alpha_z^{-1})$$

for the residue at the pole $P_z = 0$, where $\tau_1 = \tau\alpha_z^{-1} = t c_p \rho_1^{-1}$; ρ_1, ψ are the cylindrical coordinates shown in Fig.1 while the functions $F_i^{p,s}$ are determined by using relations (3.14). Note that the poles $w_1^{p,s}$ lie in the fourth quadrant, and their trajectory is shown in Fig.4.

To evaluate the residue Res_2 at the pole $P_x = -\gamma$, we find from (3.2) and (3.21):

$$w_2^{\pm p,s} = \{-i(\gamma - \tau\nu_x) \pm \nu_z \cos\varphi\, [(\gamma - \tau\nu_x)^2 + T_{p,s}^2]^{1/2}\}\, \alpha_x^{-2},$$
$$\alpha_x^2 = 1 - \nu_x^2$$

For a single-valued determination of the function we select w^+ for $\tau_2 > \tau_0$ and w^- for $\tau_2 < \tau_0$. We then obtain for the residue at the pole $P_x = -\gamma$

$$\mathrm{Res}_{2p,s}^{\pm} = F_i^{p,s} m_{p,s} [P_z T_2^{p,s} \alpha_x]^{-1} |\pm H(\tau_2 - (\beta_{p,s} - \gamma\nu_x)\alpha_x^{-1})$$
$$\tau_2 = (\tau - \gamma\nu_x)\,\alpha_x^{-1} = (t - x/v_0)\, c_p \rho_2^{-1}$$

The functions $F_i^{p,s}$ are determined by using (2.7) and the following relations:

$$P_x^{\pm} = -\gamma, \quad P_z^{\pm} = -\tau_2 \cos\eta \pm i T_2^{p,s} \sin\eta$$
$$m_{p,s}^{\pm} = \tau_2 \sin\eta \pm i T_2^{p,s} \cos\eta$$

It can be shown that the sum of the residue $\mathrm{Res}_{2p,s}^{+}$ and $J_2^{p,s}$ determined from (3.22), apart from the sign, equals the residue $\mathrm{Res}^-_{2p,s}$.

To evaluate the residue at infinity, we expand the integrand in (3.26) in a Laurent series. Carrying out this procedure for each of the functions $F_i^{p,s}$, we obtain

$$I_{x\infty}^{p,s} = \pm 2A_1 (\tau\nu_z A_1 - \gamma A_3 + 2\tau\nu_y)\, H_2^{p,s} \qquad (3.29)$$
$$I_{y\infty}^{p,s} = \{\pm 2A_3 [\tau\nu_z A_1 - (\gamma + \tau\nu_x) A_2 + \tau\nu_y] \pm 2A_3^{-1} [-\tau\nu_z A_1 - (\gamma - \tau\nu_x) A_2 + \tau\nu_y]\}\, H_2^{p,s}$$
$$I_{z\infty}^{p,s} = \pm 2A_2 [-(\gamma - \tau\nu_x) A_2 + 2\tau\nu_y]\, H_2^{p,s}$$
$$H_2^{p,s} = H(\tau - \beta_{p,s}),\ A_1 = (\nu_y\nu_z - i\nu_x)\,\alpha_z^{-2},\ A_2 = (\nu_x\nu_y + i\nu_z)\,\alpha_x^{-2},\ A_3 = -(\nu_x\nu_z + i\nu_y)\,\alpha_z^{-2}$$

Therefore, by substiuting (3.29), (3.19) and (3.15) into (3.8), omitting the tedious calculations and taking into account that the sum of the imaginary parts will be zero, we obtain

$$U_i^{p,s} = \frac{B_y c_p}{4\pi\beta_0^2} \mathrm{Re}\left[\frac{F_i^{p,s}(g_1, w)\, m_{p,s}}{\rho_1 (P_x + \gamma) T_1^{p,s}} [2H(z) H(\tau_1 - \beta_{p,s}) - H_2^{p,s}] + \frac{F_i^{p,s}(g_2, w)\, m_{p,s}}{\rho_2 P_z T_2^{p,s}} \Big|^{-} \cdot H_2^{p,s} + I_{i\infty}^{p,s} R^{-1} \right] \qquad (3.30)$$

where $I_{i\infty}^{p,s}$ are determined from relationships (3.29).

Relationships (3.30) are an exact analytic solution of the problem of the fracture of a half-space quadrant with a permanently assigned pure separation component of the displacement vector on the fault. The solution of problem (1.6) can be constructed analogously. Then by integrating relationship (3.30) with respect to time, the general solution of problem (1.3) can be represented in the form $U_1 = U_i^p + U_i^s$ $(i = x, y, z)$, where

$$\begin{aligned}
U_x^p &= A\{B_x[2\gamma^2 \operatorname{arc}_1^p - F_p] + B_y[(1+\beta_3^2)\ln_1^p + \\
&\quad {}^1/_2\,\beta_3^2\gamma\gamma_p^{-1}\operatorname{Ln}_1^p + f_p]\}H_1^p + A\{B_x[+2\gamma^2\operatorname{arc}_2^p + \Gamma_1] + \\
&\quad B_y[\Gamma_2^p + {}^1/_2\,\beta_3^2\gamma\gamma_p^{-1}\operatorname{Ln}_2^p] + B_z\Gamma_3^p\}H_2^p \\
U_x^s &= A\{B_x[-\beta_3^2\operatorname{arc}_1^s + F_s] + B_y[-\beta_3^2\ln_1^s - \\
&\quad \gamma\gamma_s\operatorname{Ln}_1^s - f_s]\}H_1^s + A\{B_x[\beta_3^2\operatorname{arc}_2^s - \Gamma_1] + \\
&\quad B_y[-\Gamma_2^s - \gamma\gamma_s\operatorname{Ln}_2^s] - B_z\Gamma_3^s\}H_2^s \\
U_y^p &= A\{B_x[(\gamma^2+\gamma_p^2)\ln_1^p + \gamma\gamma_p\operatorname{Ln}_1^p + f_p] + \\
&\quad B_y[-\beta_3^2\operatorname{arc}_1^p + F_p]\}H_1^p + A\{B_x[\Gamma_2^p + \gamma\gamma_p\operatorname{Ln}_2^p] + \\
&\quad B_y[\beta_3^2\operatorname{arc}_2^p + \Gamma_4^p] + B_z(\gamma_p^2\ln_2^p + \Gamma_5^p)\}H_2^p \\
U_y^s &= A\{B_x[-2\gamma^2\ln_1^s - {}^1/_2\,\beta_3^2\gamma\gamma_s^{-1}\operatorname{Ln}_1^s - f_s] + \\
&\quad B_y[2\gamma^2\operatorname{arc}_1^s - F_s]\}H_1^s + A\{B_x[-\Gamma_2^s - {}^1/_2\,\beta_3^2\gamma\gamma_s^{-1}\operatorname{Ln}_2^s] + \\
&\quad B_y[2\gamma^2\operatorname{arc}_2^s - \Gamma_4^s] + B_z[-\gamma^2\ln_2^s - \Gamma_5^s]\}H_2^s \\
U_z^p &= A\{B_x\Gamma_3^p + B_y[(\gamma_s^2+1)\ln_2^p + \Gamma_5^p] - B_z[\Gamma_4^p - \Gamma_1]\}H_2^p \\
U_z^s &= AB_z\beta_0^2\operatorname{arc}_1^s H_1^s + A\{-B_x\Gamma_3^s + B_y[-\gamma_s^2\ln_2^s - \Gamma_5^s] + \\
&\quad B_z[\Gamma_4^s - \Gamma_1 + \beta_0^2\operatorname{arc}_2^s]\}H_2^s
\end{aligned} \tag{3.31}$$

Here

$$\begin{aligned}
&-\Gamma_1 = \nu_y\nu_z[\tau\nu_x(3-\nu_z^2) + 2\gamma\alpha_z^2]\tau\alpha_z^{-4} \\
&-\Gamma_2^{p,s} = \tau\nu_z[\tau\alpha_x^2(2\nu_y^2 - \nu_y^2\nu_z^2 - \nu_x^2) - 2\gamma\nu_x(1+\nu_y^2)\alpha_z^2]\alpha_z^{-4}\alpha_x^{-2} - \\
&\quad 2\gamma\nu_z\alpha_x T_2^{p,s} \\
&-\Gamma_3^{p,s} = \tau(2\gamma\nu_x\nu_y - \tau\nu_y\alpha_x^2)\alpha_x^{-2} + 2\gamma\nu_y\alpha_x T_2^{p,s} \\
&\Gamma_4^{p,s} = \Gamma_1 - \frac{\tau\nu_y\nu_z\alpha_z^2[2\gamma(1+\nu_x^2) - \tau\nu_x(3-\nu_x^2)]}{\alpha_x(\nu_x^2\nu_z^2 + \nu_y^2)} + 2\nu_y\nu_z\alpha_x^{-2}\tau_2 T_2^{p,s} \\
&-\Gamma_5^{p,s} = \tau[\tau\nu_x(2\nu_y^2 - \nu_x^2\nu_y^2 - \nu_z^2) + 2\gamma(\nu_z^2 - \nu_x^2\nu_y^2)]\alpha_x^{-4} + \\
&\quad (\nu_z^2 - \nu_y^2)\alpha_x^{-2}\tau_2 T_2^{p,s} \\
&F_{p,s} = (\tau_1\sin 2\psi + 2\gamma\sin\psi)T_1^{p,s}, \quad f_{p,s} = (\tau_1\cos 2\psi + \\
&\quad 2\gamma\cos\psi)T_1^{p,s} \\
&\operatorname{arc}_1^{p,s} = \operatorname{arctg}\frac{T_1^{p,s}\sin\psi}{\gamma - \tau_1\cos\psi}, \quad \operatorname{arc}_2^{p,s} = \operatorname{arctg}(\tau_2^{-1}T_2^{p,s}\operatorname{tg}\eta) \\
&\operatorname{Ln}_1^{p,s} = \ln\frac{(\tau_1\gamma_{p,s} - \gamma T_1^{p,s})^2\sin^2\psi + [\beta_{p,s}^2 - (\tau_1\gamma - \gamma_{p,s}T_1^{p,s})\cos\psi]^2}{\beta_{p,s}^2[(\gamma - \tau_1\cos\psi)^2 + (T_1^{p,s}\sin\psi)^2]} \\
&\operatorname{Ln}_2^{p,s} = \ln\frac{(\gamma_{p,s} - T_2^{p,s}\cos\eta)^2 + \tau_2^2\sin^2\eta}{\tau_2^2 + \gamma_{p,s}^2\sin^2\eta}, \quad \ln_1^{p,s} = \ln\frac{\tau_1 + T_1^{p,s}}{\beta_{p,s}} \\
&\ln_2^{p,s} = \ln(\tau_2 - T_2^{p,s}), \; H_1^{p,s} = 2H(R\nu_z)H(\tau_1 - \beta_{p,s}) - \\
&\quad H_2^{p,s} \\
&\gamma_{p,s} = \sqrt{\gamma^2 - \beta_{p,s}^2}, \quad \beta_3^2 = 2\gamma^2 - \beta_0^2, \quad A = (4\pi\beta_0^2)^{-1}
\end{aligned} \tag{3.32}$$

4. Analysis of the results and construction of the complete solution. For the first onsets of the *P*- and *S*-waves at the point of observation, we obtain the following near-front asymptotic forms from the exact solutions (3.31) and (3.32):

$$\begin{aligned}
U^p &= A[B_x\sin 2\psi + B_y(\beta_0^2 - 2\cos^2\psi)]\frac{\sqrt{2}\sqrt{\tau_1 - 1}}{\gamma - \cos\psi} \times \\
&\quad H(z)H(\tau_1 - 1) + A[2B_x\nu_x\nu_y + B_y(2\nu_y^2 + \beta_0^2 - 2) + \\
&\quad 2B_z\nu_y\nu_z\frac{\tau - 1}{\gamma - \nu_x}H(\tau - 1)
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
U^1 &= A[2B_x\cos 2\psi + B_y\sin 2\varphi + 2B_z\sin\psi]\frac{\beta_0^{3/2}\sqrt{\tau_1 - \beta_0}}{\sqrt{2}(\gamma\beta_0^{-1} - \cos\psi)} \times \\
&\quad H(z)H(\tau_1 - \beta_0) + A[B_x(\nu_x^2 - 4\nu_x^2\nu_y^2 + \nu_y^2)^{1/2} + 2B_y\nu_y \times \\
&\quad \sqrt{1 - \nu_y^2} + B_z(\nu_y^2 - 4\nu_y^2\nu_z^2 + \nu_z^2)^{1/2}] \times \\
&\quad \frac{\beta_0^2(\tau - \beta_0)}{(\gamma - \beta_0\nu_x)\nu_z}H(\tau - \beta_0)
\end{aligned}$$

The factors in front of the unit functions $H_1^{p,s}$ in the exact solution (3.31) and the first components in the asymptotic solutions (4.1) agree with the exact solutions of the plane problem /1/ and the asymptotic solutions /7/, respectively. This part of the general solution is cylindrical waves that act in the domain $z > 0$ and fall off inversely as the square roots of the distance. The remaining part of the solution is spherical waves produced by the angular points. The spherical waves exist in all space, but they damp out considerably more rapidly with distance than the cylindrical waves, namely, in inverse proportion to the distance. Therefore, the greatest part of the information about the source will be contained, for large R, in the cylindrical waves while the solutions of plane problems /1, 6, 7/ can be used effectively to analyse them.

Radiation directivity patterns of cylindrical P-waves (upper series) and S-waves (lower series) are presented in Fig.5 for a rate $v_0 = 0{,}6C_s$ of ripping open the fault and different values of the angle that governs the magnitude of the separation and shear components of the displacement vector at the fault $\psi_0 = \mathrm{arctg}\,(B_y/B_x)$, $\psi_0 = 0, 20, 40, 90°$ (patterns a-d). For a pure shear fault $\psi_0 = 0$ (Fig.5a), the presence of two mutually perpendicular nodal planes is characteristic (for both the P- and the S-waves) in which there are no slips and where the sign of the displacement vector changes as it goes through them.

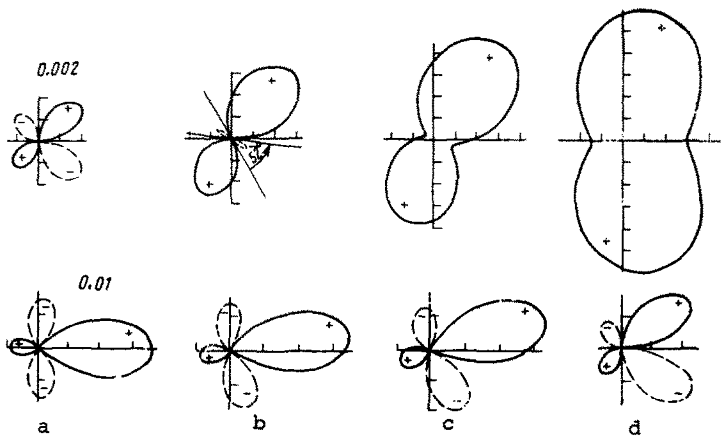

Fig.5

For a pure cleavage fault $\psi_0 = 90°$ (Fig.5d), the seismic P-wave radiation is positive in all directions, i.e., compressive. As is shown in Figs.5b and c, for a complex fault the radiation pattern will depend on the ratio between the normal and tangential components of the displacement vector at the fault. As the separation component of the displacement vector grows at the fault, the angle between the two nodal planes will decrease in the P-waves, will equal zero for $\psi_0 = 30°$, and for $\psi_0 > 30°$ the first onset of P-waves in all directions will have identical sign, i.e., the quadrant-by-quadrant distribution of signs of the first onset of longitudinal waves will be absent. Rotation of the nodal planes by 45° will occur for the S-waves as the angle ψ increases from 0 to 90°. For a pure cleavage fault, the radiation pattern of seismic radiation will be symmetrical for S-waves and asymmetrical in the presence of a shear component. These radiation pattern singularities of seismic radiation can underlie the processing of seismological observations for the isolation of complex foci of tectonic earthquakes.

Therefore, the correct determination of the magnitudes of the cleavage and shear components of the displacement vector for a complex fault enables the accuracy of determining the mechanisms and other dynamical parameters of foci of large-scale tectonic earthquakes to be improved.

It should be noted that solutions (3.31) and (3.32) were obtained under the condition that the vector $\mathbf{B} = \mathrm{const}$. On the basis of the linearity of the fundamental equations, the solution for an arbitrary dependence of the vector $\mathbf{B}$ on the time $\mathbf{B} = \mathbf{B}(t)$ can be obtained by using the Duhamel integral

$$\int_0^t B(t-\tau)\, U^{p,s}(x, y, z, \tau)\, d\tau \tag{4.2}$$

where $U^{p,s}$ is the solution defined by relations (3.43) and (3.44).

Solutions (3.31) and (3.32) are exact analytic solutions of the problem of complex fracture (shear with separation) of a quadrant of space under the condition that the fracture front propagates at a constant velocity. On the basis of these solutions and because of the linearity of the fundamental equations, the solution for a rectangular fracture area, one of whose edges propagates at a constant velocity v_0, is constructed as follows.

Let $U^{p,s}(x, y, z, v_0, t)$ be the complete displacement vector that is constructed on the basis of solutions (3.31) and (3.32). Then the solution for a rectangular fracture area whose width is W_0 and whose length is $L = v_0t$, will have the form

$$U_1^{p,s} = U^{p,s}(x, y, z + {}^1/_2W_0, v_0t) - U^{p,s}(x, y, z - {}^1/_2W_0, v_0, t) \qquad (4.3)$$

The solution of the problem taking account of the arrest of the moving edge of the fault at the time $t = t_0$ is constructed in the same way at the solution of the plane problems in /1, 6, 7/ and has the form

$$U_2^{p,s} = U_1^{p,s}(x, y, z, W_0, v_0, t) - U_1^{p,s}(x - v_0t_0, y, z, W_0, v_0, t - t_0) \qquad (4.4)$$

Using the superposition principle and the fundamental solution (4.4), we can construct a general solution of the problem of an arbitrary system of complex curvilinear faults propagating at variable velocities and transfer to a detailed quantitative analysis of high-frequency seismic radiation.

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